H_∞-Constrained Quasi-Linear Quadratic Gaussian Control with Loop Transfer Recovery

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In this paper we propose a new nonlinear controller design method, called quasi-linear quadratic Gaussian/H-infinity/loop transfer recovery (QLQG/H_∞/LTR), for nonlinear multivariable systems with hard nonlinearities such as Coulomb friction and dead-zones. We consider H_∞-constraints for the optimization of statistically linearized systems, by replacing the covariance Lyapunov equation by a modified Riccati equation, whose solution leads to an upper bound QLQG performance. As a result, the nonlinear correction term is included in the Riccati equation which, in general, is excessively difficult to solve numerically. To solve this problem, we use the modified loop shaping technique and derive analytic proofs of the LTR condition. Finally, the H_∞-constrained nonlinear controller is synthesized by an inverse random input describing function technique (IRIDF). The proposed design method for a hard nonlinear multivariable systems has better robustness to unstructured uncertainty and hard nonlinearities than the QLQG/LTR method. A flexible link system with Coulomb frictions serves as a design example for our methodology.

Key Words: QLQG, QLQG/H_∞/LTR, QLQG/LTR, RIDF, IRIDF

1. Introduction

For the multivariable systems with hard nonlinearities such as Coulomb friction, dead-zones and backlash, linear multivariable control methods are limited in their applicability due to the discontinuous differentiability of the nonlinearities. It is known that the use of statistical linearization techniques can be effective for many hard nonlinear systems. There is, however, no general and unified control methodology for hard nonlinear systems. If we can apply the systematic multivariable control methods of the linear systems can be applied to hard nonlinear systems under any acceptable conditions, it is very desirable to develop the general nonlinear controller design methods.

For the above problem, Beaman (1984) developed the quasi-linear quadratic Gaussian control method (QLQG), which combines optimal estimation and control for statistically linearized systems. This method, however, has the draw back that selecting design parameters can be complex and the nonlinear correction term is often quite complicated. In addition, QLQG control does not fully address performance and stability robustness issues. In order to solve these and other issues, the quasi-linear quadratic Gaussian control with loop transfer recovery (QLQG/LTR) has been developed in (Kim, 1987, 1989, 1994), which has an LQG (H_2) performance criterion.

In this paper, we propose the QLQG/H_∞/LTR method, which extends the earlier QLQG/LTR method. The proposed QLQG/H_∞/LTR method employs similar design techniques as used in the QLQG/LTR method, such as random input describing functions (RIDF) (Gelb and Vander

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Velde, 1968), IRIDF (Suzuki and Hedrick, 1985) and modified LTR. However, the difference is that the minimization cost of the QLQG/H_∞/LTR derives from mixed LQG and H_∞ control (Bernstein and Haddad, 1989). Therefore, our proposed method can offer both stability-robustness (in the sense of H_∞-norm bound) and nominal performance (in the sense of LQG cost bound) as well as robustness to hard nonlinearties. Our method is a general version of the QLQG/LTR method, since if the H_∞-norm parameter γ approaches infinity, then QLQG/H_∞/LTR control becomes identical to QLQG/LTR. As a design example, we consider a flexible link system containing Coulomb friction is to examine robustness of the controller to the neglected elastic modes and nonlinear effects.

2. QLQG/H_∞/LTR Control

The dynamic equation for hard nonlinear systems is

$$\dot{\boldsymbol{x}}(t)=\boldsymbol{f}(\boldsymbol{x}(t))+\boldsymbol{B}\boldsymbol{u}(t)+\boldsymbol{D}_1\boldsymbol{w}_1(t) \tag{1}$$

where $\boldsymbol{x}(t)\in\boldsymbol{R}^n$ is the plant state vector, $\boldsymbol{f}(\boldsymbol{x}(t))\in\boldsymbol{R}^n$ is the nonlinear dynamic vector, $\boldsymbol{u}(t)\in\boldsymbol{R}^m$ is the control input vector, and $\boldsymbol{w}_1(t)\in\boldsymbol{R}^p$ is the plant disturbance vector. When the hard nonlinearities of the above nonlinear system Eq. (1) are symmetric, memoryless and single-valued, we can statistically approximate these by describing functions (DF). The nth-order stabilizable and detectable plant and weighted errors are given by

$$\begin{aligned}&\dot{\boldsymbol{x}}(t)=\boldsymbol{N}(\sigma_x)+\boldsymbol{B}\boldsymbol{u}(t)+\boldsymbol{D}_1\boldsymbol{w}_1(t)\\&\boldsymbol{z}_1(t)=\boldsymbol{E}_1(t)\boldsymbol{x}(t),\ \boldsymbol{z}_2(t)=\boldsymbol{E}_2(t)\boldsymbol{u}(t)\\&\boldsymbol{z}_{1\infty}(t)=\boldsymbol{E}_{1\infty}(t)\boldsymbol{x}(t),\ \boldsymbol{z}_{2\infty}(t)=\boldsymbol{E}_{2\infty}(t)\boldsymbol{u}(t)\\&\boldsymbol{y}(t)=\boldsymbol{C}\boldsymbol{x}(t)+\boldsymbol{D}_2\boldsymbol{w}_2(t)\end{aligned} \tag{2}$$

where $\boldsymbol{N}(\sigma_x)$ is the $(n\times n)$ statistically linearized plant and σ_x is the standard deviation of the plant states. Then the n_c-order nonlinear dynamic controller is given by

$$\begin{aligned}&\dot{\boldsymbol{z}}(t)=\boldsymbol{N}_z(\sigma_z)\boldsymbol{z}(t)+\boldsymbol{H}\boldsymbol{y}(t)\\&\boldsymbol{u}(t)=\boldsymbol{G}\boldsymbol{z}(t)\end{aligned} \tag{3}$$

where $\boldsymbol{N}(\sigma_z)$ is the $(n_c\times n_c)$ statistically linearized controller matrix, and σ_z is the standard deviation of the controller states, such that the following design criteria are satisfied:

1) the closed-loop system Eq. (2) is asymptotically stable;

2) the $q_\infty\times p$ closed-loop transfer function matrix

$$\boldsymbol{H}_N(s)=\tilde{\boldsymbol{E}}_\infty(s\tilde{\boldsymbol{I}}_{\tilde{n}}-\tilde{\boldsymbol{N}}(\sigma))^{-1}\tilde{\boldsymbol{D}} \tag{4}$$

from $\tilde{\boldsymbol{w}}(t)$ to $\tilde{\boldsymbol{z}}_\infty$ satisfies the constraint

$$\|\boldsymbol{H}_N\|_\infty\le\gamma \tag{5}$$

where γ is a given positive constant,

$$\tilde{\boldsymbol{w}}(t)=[\boldsymbol{w}_1(t),\ \boldsymbol{w}_2(t)]^T,\ \tilde{\boldsymbol{z}}_\infty(t)=[\boldsymbol{z}_{1\infty}(t),\ \boldsymbol{z}_{2\infty}(t)]^T$$

$$\tilde{\boldsymbol{N}}=\begin{bmatrix}\boldsymbol{N}(\sigma_x) & \boldsymbol{B}\boldsymbol{G}\\ \boldsymbol{H}\boldsymbol{C} & \boldsymbol{N}_z(\sigma_z)\end{bmatrix},\tilde{\boldsymbol{D}}=\begin{bmatrix}\boldsymbol{D}_1 & 0\\ 0 & \boldsymbol{H}\boldsymbol{D}_2\end{bmatrix},\tilde{\boldsymbol{E}}_\infty=\begin{bmatrix}\boldsymbol{E}_{1\infty} & 0\\ 0 & \boldsymbol{E}_{2\infty}\end{bmatrix}$$

3) the performance functional

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{N}_z(\sigma_z),\ \boldsymbol{H},\ \boldsymbol{G})&=\lim_{t\to\infty}\boldsymbol{E}[\boldsymbol{x}^T(t)\boldsymbol{R}_1\boldsymbol{x}(t)\\&\quad+\boldsymbol{u}^T(t)\boldsymbol{R}_2\boldsymbol{u}(t)]\\&=tr(\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{R}})\end{aligned} \tag{6}$$

where $\tilde{\boldsymbol{Q}}(=\lim_{t\to\infty}\boldsymbol{E}[\tilde{\boldsymbol{x}}(t)\tilde{\boldsymbol{x}}^T(t)])$ is the steady-state closed-loop state covariance, satisfies the $(\tilde{n}\times\tilde{n})$ algebraic Lyapunov equation

$$\tilde{\boldsymbol{N}}\tilde{\boldsymbol{Q}}+\tilde{\boldsymbol{Q}}\tilde{\boldsymbol{N}}^T+\tilde{\boldsymbol{V}}=0 \tag{7}$$

where $\tilde{\boldsymbol{V}}=\begin{bmatrix}\boldsymbol{V}_1 & 0\\ 0 & \boldsymbol{H}\boldsymbol{V}_2\boldsymbol{H}^T\end{bmatrix}$, $\boldsymbol{V}_1=\boldsymbol{D}_1\boldsymbol{D}_1^T$, $\boldsymbol{V}_2=\boldsymbol{D}_2\boldsymbol{D}_2^T$.

$\tilde{\boldsymbol{V}}$ is the power spectral density of the Gaussian white noise input, and

$$\tilde{\boldsymbol{R}}=\begin{bmatrix}\boldsymbol{R}_1 & 0\\ 0 & \boldsymbol{G}^T\boldsymbol{R}_2\boldsymbol{G}\end{bmatrix},\ \boldsymbol{R}_1=\boldsymbol{E}_1^T\boldsymbol{E}_1,\ \boldsymbol{R}_2=\boldsymbol{E}_2^T\boldsymbol{E}_2,$$

$\tilde{\boldsymbol{R}}$ is the control weighting matrix.

The state equation of the closed-loop system can then be written as

$$\dot{\tilde{\boldsymbol{x}}}(t)=\tilde{\boldsymbol{N}}\tilde{\boldsymbol{x}}(t)+\tilde{\boldsymbol{D}}\tilde{w}(t) \tag{8}$$

where $\tilde{\boldsymbol{x}}(t)=[\boldsymbol{x}(t),\ \boldsymbol{z}(t)]^T$.

2.1 QLQG with the constraint of H_∞ disturbance attenuation

The key of the minimization issue in 3) is to satisfy the following result (Bernstein and Haddad, 1989). Let $(\boldsymbol{N}_z(\sigma_z),\ \boldsymbol{H},\ \boldsymbol{G})$ be given and $\tilde{\boldsymbol{Q}}_\infty\in\boldsymbol{R}^{\tilde{n}\times\tilde{n}}$ a nonnegative matrix satisfying the algebraic Riccati equatiin

$$\tilde{\boldsymbol{N}}\tilde{\boldsymbol{Q}}_\infty+\tilde{\boldsymbol{Q}}_\infty\tilde{\boldsymbol{N}}^T+\gamma^{-2}\tilde{\boldsymbol{Q}}_\infty\tilde{R}_\infty\tilde{\boldsymbol{Q}}_\infty+\tilde{\boldsymbol{V}}=0 \tag{9}$$

where

$$\tilde{R}_\infty=\begin{bmatrix} R_{1\infty} & 0 \\ 0 & G^T R_{2\infty} G \end{bmatrix},\ R_{1\infty}=E_{1\infty}^T E_{1\infty}$$

$$R_{1\infty}=E_{1\infty}^T E_{1\infty}=\beta^2 R_2,$$

and β is a nonnegative constant. Then $(\tilde{N}, [\gamma^{-2}\tilde{Q}\tilde{R}_\infty\tilde{Q}])$ is stabilizable if and only if $\tilde{N}$ is asymptoically stable. The stability of $\tilde{N}$ implies that $\|H_N\|_\infty \le \gamma$ and $\tilde{Q}\le\tilde{Q}_\infty$. Consequently.

$$J(N_z(\sigma_z), H, G)\le J_\infty(N_z(\sigma_z), H, G, \tilde{Q}_\infty)$$

where

$$J_\infty(N_z(\sigma_z), H, G, \tilde{Q}_\infty)=tr(\tilde{Q}_\infty\tilde{R}_\infty) \quad (10)$$

$N_z(\sigma_z)$, H and G for the system Eq. (2) should be determined such that Eq. (9) has a nonnegative solution $\tilde{Q}_\infty$ that minimizes $J_\infty(N_z(\sigma_z), H, G, \tilde{Q}_\infty)$. To optimize Eq. (10) subject to Eq. (9), the left-hand side of Eq. (10) can be augmented to $tr(\tilde{Q}_\infty\tilde{R}_\infty)$ and represented as the auxiliary cost via a symmetric Lagrange multiplier matrix $\tilde{P}$ to form

$$L=tr\{\tilde{Q}_\infty\tilde{R}_\infty+[\tilde{N}\tilde{Q}_\infty+\tilde{Q}_\infty\tilde{N}^T +\gamma^{-2}\tilde{Q}_\infty\tilde{R}_\infty\tilde{Q}_\infty+\tilde{V}]\tilde{P}\} \quad (11)$$

where

$$\tilde{Q}_\infty=\begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix},\ \tilde{P}=\begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}$$

Then differentiation of L with respect to the submatrices of $\tilde{Q}_\infty$, $\tilde{P}$ and parameters of controller $(N_z(\sigma_z), H, G)$ yields nine equations. By manipulating these equations the following results can be obtained:

Riccati equations:

$$N\bar{Q}+\bar{Q}N^T+\gamma^{-2}\bar{Q}R_{1\infty}\bar{Q}-\bar{Q}C^T V_2^{-1} C\bar{Q} + V_1=0 \quad (12)$$

$$(N+\gamma^{-2}[\bar{Q}+\hat{Q}]R_{1\infty})^T\bar{P}+\bar{P}(N+\gamma^{-2}[\bar{Q}+\hat{Q}]R_{1\infty})+R_1-S^T\bar{P}BR_2^{-1}B^T\bar{P}S +\Psi(\bar{P}, \bar{Q}, \hat{Q}, N)=0 \quad (13)$$

$$(N-BR_2^{-1}B^T\bar{P}S+\gamma^{-2}\bar{Q}R_{1\infty})\hat{Q} +\hat{Q}(N-BR_2^{-1}B^T\bar{P}S+\gamma^{-2}\bar{Q}R_{1\infty})^T +\gamma^{-2}\hat{Q}(R_{1\infty}+\beta^2 S^T\bar{P}BR_2^{-1}B^T\bar{P}S)\hat{Q} +\bar{Q}C^T V^{-2} C\bar{Q}=0 \quad (14)$$

where

$$S=(I_n+\beta^2\gamma^{-2}\hat{Q}\bar{P})^{-1},\ \Psi(\bar{P}, \bar{Q}, \hat{Q}, N(\sigma_x)) =2tr\left[\bar{P}\frac{\partial N(\sigma_z)}{\partial(\bar{Q}+\hat{Q})}(\bar{Q}+\hat{Q})\right]$$

Controller parameters:

$$N_z(\sigma_z)=N-BR_2^{-1}B^T\bar{P}S-\bar{Q}C^T V_2^{-1}C +\gamma^{-2}\bar{Q}R_{1\infty} \quad (15)$$

$$H=\bar{Q}C^T V_2^{-1} \quad (16)$$

$$G=-R_2^{-1}B^T\bar{P}S \quad (17)$$

Note that these results are the counterpart version of the linear case (Bernstein and Haddad, 1989) for a statistically linearized system, except for the DF and nonlinear correction terms in Eq. (13). Equations (12) and (13) are similar to the filter and regulator Riccati equations of LQG theory, while Eq. (14) has no counterpart in the standard theory. Equations (12) can be solved independently, but Eqs. (13) and (14) must be solved simultaneously, since these equations are coupled. Furthermore, Eq. (13) contains nonlinear correction terms $\Psi(\cdot)$, for which solutions are very difficult to obtain. If the nonlinear correction term can be neglected, the controller for hard nonlinear systems can be easily designed similar to linear systems. Thus, we will show that LTR techniques for statistically linearized systems provide another advantage by eliminating the nonlinear correction term in Eq. (13).

2.2 Design of the target filter loop

To develop our loop shaping problem, without loss of generality we make some assumptions commonly done in the optimalcontrol problems. A fictious process and measurement white noise is considered for loop shaping, and modified Kalman filter frequency domain equality (KFDE)is also used. The statistically linearized design plant is

$$\begin{aligned} x(t)&=Nx(t)+Bu(t)+Lw_1(t)\\ y(t)&=Cx(t)+D_2w_2(t)\end{aligned} \quad (18)$$

where

$E[w_1(t)]=0$, $E[w_2(t)]=0$, $E[w_1(t)w_1^T(t+\tau)]=I\delta(\tau)$, $E[w_2(t)w_2^T(t+\tau)]=I\delta(\tau)$, $V_1=D_1D_1^T=LL^T$, $V_2=D_2D_2^T=\mu I$, μ=a positive constant as a design parameter.

Weightings for estimated errors and control are defined as

$R_1=C^TC$, $R_2=\rho I$, $R_{1\infty}=C^TC/\mu$, $\beta=0$, $R_{2\infty}=0$, $S=I_n$, $\alpha^{-2}=1-\gamma^{-2}$, $1<\gamma<\infty$, $1<\alpha<\infty$, ρ=a

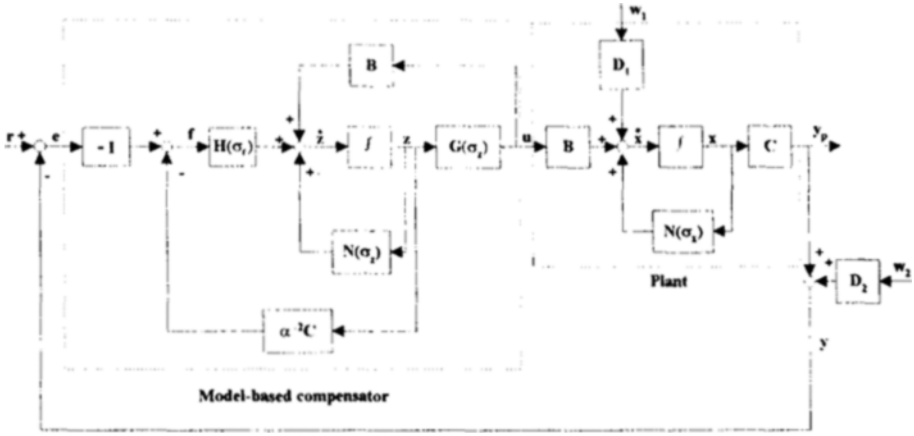

Fig. 1 The structure of the QLQG/H$_\infty$/LTR control system

positive constant as a design parameter.

Then the Riccati equations can be expressed as ollwos:

$$N\bar{Q}+\bar{Q}N^T+LL^T-\frac{\alpha^{-2}}{\mu}\bar{Q}C^TC\bar{Q}=0 \quad (19)$$

$$(N+\frac{\gamma^{-2}}{\mu}[\bar{Q}+\hat{Q}]C^TC)^T\bar{P}+\bar{P}(N+\frac{\gamma^{-2}}{\mu}[\bar{Q}+\hat{Q}]C^TC)+C^TC-\frac{1}{\rho}\bar{P}BB^T\bar{P}+\Psi(\bar{P},\ \bar{Q},\ \hat{Q},\ N)=0 \quad (20)$$

$$(N-\frac{1}{\rho}BB^T\bar{P}+\frac{\gamma^{-2}}{\mu}\bar{Q}C^TC)\hat{Q}+\hat{Q}(N-\frac{1}{\rho}BB^TP+\frac{\gamma^{-2}}{\mu}QC^TC)^T+\frac{\gamma^{-2}}{\mu}\hat{Q}C^TC\hat{Q}+\frac{1}{\mu}\bar{Q}C^TC\bar{Q}=0 \quad (21)$$

Because the Principle of Separation holds in one direction, Eq. (19) corresponds to the Riccati equation to obtain a filter gain that is broken at the plant output. The target filter loop function matrix $G_{KF}(s)$ is written as

$$G_{KF}(s)=C(sI-N)^{-1}H \quad (22)$$

Then, the modiied KFDE can be derived in a similar manner. If the statistically linearized plant is given by Eq. (18), the modified KFDE can be obtained as follows:

$$[I+\alpha^{-2}G_{KF}(s)][I+\alpha^{-2}G_{KF}(-s)]^T=I+\frac{\alpha^{-2}}{\mu}G_{FOL}(s)G_{FOL}^T(s) \quad (23)$$

$$[I+\alpha^{-2}G_{KF}(jw)][I+\alpha^{-2}G_{KF}(-jw)]^T=I+\frac{\alpha^{-2}}{\mu}[C\phi(jw)L][C\phi(-jw)L]^T \quad (24)$$

The above equation can be represented as the singular value of the matrix.

$$\sigma_i[I+\alpha^{-2}G_{KF}(jw)]=\sqrt{1+\frac{1}{\mu}\sigma^2{}_i[\alpha^{-1}C\phi(jw)L]} \quad (25)$$

Thus, except for the neighborhood of the cross-over frequency, Eq. (25) can be expressed approximately as

$$\sigma_i[G_{KF}(jw)]\cong\frac{1}{\sqrt{\mu}}\sigma_i[\alpha C\phi(jw)L \quad (26)$$

Equation (26) implies that if we select L such that $\alpha C\phi(jw)L$ is the target loop shape, then when the Doyle-Stein condition is satisfied, the recovered loop shape approximates the target loop shape in the frequency range of interest. Dropping the last term in Eq. (23), a modified KFDE can be established:

$$[I+\alpha^{-2}G_{KF}][I+\alpha^{-2}G_{KF}(-s)]^T\geq I \quad (27)$$

From Eq. (26), we obtain

$$G_{KF}(s)\cong\frac{\alpha}{\sqrt{\mu}}C(sI-N)^{-1}L \quad (28)$$

The filter gain H can be obtained from Eqs. (16) and (19)

$$H=\frac{1}{\mu}\bar{Q}C^T \quad (29)$$

Solving the remaining equations, the other controller dynamics are

$$N_z(\sigma_z) = \boldsymbol{N} - \alpha^{-2}\boldsymbol{HC} + \boldsymbol{BG} \tag{30}$$

$$\boldsymbol{G} = -\frac{1}{\rho}\boldsymbol{B}^T\boldsymbol{P} \tag{31}$$

Fig. 1 shows the structure of the QLQG/H$_\infty$/LTR control system.

2.3 LTR using cheap control

In order to design the control gain, we introduce the cheap control problem. With this problem, the nonlinear correction term can be treated by examining the behavior of the control algebraic Riccati equation (CARE), Eq. (20). First, we check the order of the magnitude of each term in Eq. (20) as ρ approaches 0. $\|\Psi\|$ is approximately $\|\bar{\boldsymbol{P}}(\boldsymbol{N}+\gamma^{-2}/\mu[\bar{\boldsymbol{Q}}+\hat{\boldsymbol{Q}}])\boldsymbol{C}^T\boldsymbol{C}\|$, and $\|\boldsymbol{C}^T\boldsymbol{C}\|$ and $\|\boldsymbol{BB}^T\|$ are both finite. Then the order of the magnitude for $\|\Psi\|$ should be checked, and the following conditions, which guarantee good LTR, should also be satisfied:

1) $\|\boldsymbol{C}^T\boldsymbol{C}\| \cong \|\bar{\boldsymbol{P}}\boldsymbol{BB}^T\bar{\boldsymbol{P}}\|/\rho \le \|\boldsymbol{BB}^T\|\cdot\|\bar{\boldsymbol{P}}\|^2/\rho$ (32)

or

$$\|\bar{\boldsymbol{P}}\| \ge (\rho\|\boldsymbol{C}^T\boldsymbol{C}\|(\|\boldsymbol{BB}^T\|)^{-1})^{1/2} \tag{33}$$

2) $\|\bar{\boldsymbol{P}}(\boldsymbol{N}+\gamma^{-2}/\mu[\bar{\boldsymbol{Q}}+\hat{\boldsymbol{Q}}])\boldsymbol{C}^T\boldsymbol{C}\| \le \|\bar{\boldsymbol{P}}\|\cdot \|(\boldsymbol{N}+\gamma^{-2}/\mu[\bar{\boldsymbol{Q}}+\hat{\boldsymbol{Q}}])\boldsymbol{C}^T\boldsymbol{C}\| \ll \|\boldsymbol{C}^T\boldsymbol{C}\|$ (34)

or

$$\|(\boldsymbol{N}+\gamma^{-2}/\mu[\bar{\boldsymbol{Q}}+\hat{\boldsymbol{Q}}])\boldsymbol{C}^T\boldsymbol{C}\| \ll \|\boldsymbol{C}^T\boldsymbol{C}\|\cdot\|\bar{\boldsymbol{P}}\|^{-1} = (\|\boldsymbol{BB}^T\|\cdot\|\boldsymbol{C}^T\boldsymbol{C}\|/\rho)^{1/2} \tag{35}$$

From Eq. (35) the LTR index (q) can be defined in the following form:

$$q = \|(\boldsymbol{N}+\gamma^{-2}/\mu[\bar{\boldsymbol{Q}}+\hat{\boldsymbol{Q}}])\boldsymbol{C}^T\boldsymbol{C}\|\cdot (\rho\cdot(\|\boldsymbol{BB}^T\|\cdot\|\boldsymbol{C}^T\boldsymbol{C}\|)^{-1})^{1/2} \ll 1 \tag{36}$$

For the scalar case ($\boldsymbol{B}=\boldsymbol{C}=\boldsymbol{L}=\boldsymbol{1}$),

$$q = \|(N+\gamma^{-2}/\mu[Q+\hat{Q}]\cdot\|\rho^{1/2} \ll 1 \tag{37}$$

and the parameters of the controller are

$$N_z = N - \alpha^{-2}H + G,\ H = \bar{Q}/\mu,\ G = -\bar{P}/\rho \tag{38}$$

Then, the order of the magnitude for $\|\Psi\|$ is equivalent to that of the LTR index (q) as $\rho \to 0$ for the simple scalar case. Let us examine the limit behavior of the LTR index corresponding to a simplifying of this problem. A scalar variance and filter gain can be calculated from Eqs. (19) and (38)

$$\bar{Q} = \mu\alpha^2(N+\sqrt{N^2+1/u}) \tag{39}$$

$$H = \alpha^2(N+\sqrt{N^2+1/u}) \tag{40}$$

Equations (20) and (21) can be written

$$\bar{P}^2 - 2\rho(N(1+k)+\gamma^{-2}[\bar{Q}+\hat{Q}]/\mu)\bar{P} - \rho = 0 \tag{41}$$

$$\gamma^{-2}\hat{Q}/\mu + 2(N-\bar{P}/\rho+\gamma^{-2}\hat{Q}/\mu)\hat{Q} + \bar{Q}^2/\mu = 0 \tag{42}$$

where k is a constant which depends on the nonlinearity. For example, $k=1$ if $f(x)=x^3$ and $k=-1/2$ if $f(x)=sgn(x)$. But because Eqs. (41) and (42) are coupled, the control gain cannot be directly calculated in this form. To address this coupled problem, we propose the following iteration algorithm;

Setp 1: Take initial Values for $\bar{Q}$, $\bar{P}$ as $\gamma\to\infty$ (QLQG case). Then

$$\bar{Q} = \mu(N+\sqrt{N^2+1/u}) \tag{43}$$

$$\bar{P} = \rho N(1+k) + \sqrt{(\rho N(1+k))^2+\rho} \tag{44}$$

Step 2: Replace $\bar{Q}$, $\bar{P}$ into Eq. (41) with $\gamma\to\infty$. Then

$$\hat{Q} = (N+\sqrt{N^2+1/\mu})\cdot [2(Nk+\sqrt{(N(1+k))^2+1/\rho})]^{-1} \tag{45}$$

Step 3: Replace $\hat{Q}$. $\bar{Q}$ into Eq. (41) again for a certain $\gamma(\neq\infty)$. Then

$$\bar{P} = \rho(N(1+k)+[\bar{Q}+\hat{Q}]/\mu) + \sqrt{(\rho^2(N(1+k)+\gamma^{-2}[\bar{Q}+\hat{Q}])^2/\mu+\rho} \tag{46}$$

The control gain is then given by

$$G = -[N(1+k)+\gamma^{-2}(N+\sqrt{N^2+1/\mu})+\frac{\gamma^{-2}(N+\sqrt{N^2+1/\mu})^2}{2\mu(Nk+\sqrt{(N(1+k))^2 1/\rho})}] - \sqrt{N(1+k)+\gamma^{-2}(N+\sqrt{N^2+1/\mu})+\frac{\gamma^{-2}(N+\sqrt{N^2+1/\mu})^2}{2\mu(Nk+\sqrt{(N(1+k))^2+1/\rho})}+1/\rho} \tag{47}$$

Now let us check the order of magnitude of H and G, respectively.

1) $|N| \gg 1/\sqrt{u}$ and $|N| \gg 1/\sqrt{\rho}$

$$o(H) = o(2\alpha^2 N),\ o(G) = o(2N+4\gamma^{-2}N) \tag{48}$$

2) $|N| \ll 1/\sqrt{u}$ and $|N| \ll 1/\sqrt{\rho}$ $(1/\sqrt{u} \ll 1/\sqrt{\rho})$

$$o(H) = o(\alpha^2\sqrt{u}),\ o(G) = o(1/\sqrt{\rho}) \tag{49}$$

From Eq. (9), the weighted scalar variance can be calculated as

$$\begin{aligned}&\gamma^{-2}(\bar{Q}+\hat{Q})/\mu \\ &= -(N(2N+2G-\alpha^{-2}H)-GH)/(2N \\ &\quad +2G-\alpha^{-2}H)+\{(N(2N+2G \\ &\quad -\alpha^{-2}H)-GH)/(2N+2G-\alpha^{-2}H))^2 \\ &\quad -\gamma^{-2}V_1/\mu\}^{1/2}\end{aligned} \tag{50}$$

Finally, checking the different orders of magnitude for the LTR index, we show that the nonlinear correction term in the Riccati equation can be neglected, and that the modified coupled Riccati equation is the same as the linear one:

Case 1: $|N| \gg 1/\sqrt{\mu}$ and $|N| \gg 1\sqrt{\rho}(q \gg 1)$

$$\begin{cases} o(H) = o(2\alpha^2 N) \\ o(G) = o(2N+4\gamma^{-2}N) \\ o(\gamma^{-2}/\mu(\bar{Q}+\hat{Q})) = o(0) \end{cases}$$

Since $o(q(=\|N\|)) \ll 1/\sqrt{\rho}$, if $\rho \to 0$ the given condition $|N| \gg 1/\sqrt{\rho}$ is not satisfied. Therefore, when $\rho \to 0$ the possibility of $\rho \cong 1$ does not exist.

Case 2: $|N| \cong 1/\sqrt{\rho}$ $(q \cong 1)$

$$\begin{cases} o(H) = o(2\alpha^2 N) \\ o(G) = o(2N+4\gamma^{-2}N) \\ o(\gamma^{-2}/\mu(\bar{Q}+\hat{Q})) = o(0) \end{cases}$$

Since $o(q(=\|N\|)) \cong 1$, if $\rho \to 0$, the given condition $|N| \cong 1/\sqrt{\rho}$ is not satisfied. Therefore, when $\rho \to 0$, the possibility of $q \cong 1$ does not exist.

Case 3: $1/\sqrt{u} \ll |N| \ll 1/\sqrt{\rho}$ $(q \ll 1)$

$$\begin{cases} o(H) = o(2\alpha^2 N) \\ o(G) = o(1/\sqrt{\rho}) \\ o(\gamma^{-2}/\mu(\bar{Q}+\hat{Q})) = o(0) \end{cases}$$

If $\rho \to 0$, the given condition $|N| \ll 1/\sqrt{\rho}$ is satisfied. Therefore, when $\rho \to 0$, the condition $q \ll 1$ is satisfied.

Case 4: $|N| \ll 1/\sqrt{u} \ll 1/\sqrt{\rho}$ $(q \ll 1)$

$$\begin{cases} o(H) = o(\alpha^2\sqrt{u}) \\ o(G) = o(1/\sqrt{\rho}) \\ o(\gamma^{-2}/\mu(\bar{Q}+\hat{Q})) = o(1/\sqrt{u}[\alpha^2/2 \\ \quad +\sqrt{\alpha^4/4-\gamma^{-2}V_1}]) \end{cases}$$

If α, V_1 and μ are finite, then $\gamma^{-2}/\mu(\bar{Q}+\hat{Q})$ is finite in the stable system and $o(q(=\|N+\gamma^{-2}/\mu(\bar{Q}+\hat{Q})\|)) \ll 1\sqrt{\rho}$. If $\rho \to 0$, the given condition $|N| \ll 1/\sqrt{\rho}$ is satisfied and $q \ll 1$ is also satisfied.

As $\|N\|$ is finite in the stable system with finite inputs, q approaches 0 as ρ approaches 0. This means that the LTR conditions for the statistically linearized system are the same as for the linear system. Therefore, the magnitude o the nonlinear correction term $\|\Psi\|$ is the order of q. When the LTR conditions are satisfied, the correction term can be neglected and the modified coupled Riccati equations are of the same form as the linear case.

Finally, we now derive the limiting behavior of the loop transfer function matrix at the plant output. From Eq. (20), if the LTR condition is satisfied, the limiting behavior of Eq. (21) as $\rho \to 0$ is

$$\boldsymbol{C}^T\boldsymbol{C} - (\boldsymbol{PB}/\sqrt{\rho}) \cdot (\boldsymbol{B}^T\boldsymbol{P}/\sqrt{\rho}) \to \boldsymbol{0} \tag{51}$$

Substituting the control gain $\boldsymbol{G} = -\boldsymbol{B}^T\boldsymbol{P}/\rho$ into Eq. (51),

$$(\sqrt{\rho}\,\boldsymbol{G})^T(\sqrt{\rho}\,\boldsymbol{G}) \to \boldsymbol{UC} \tag{52}$$

This implies that

$$\lim_{\rho \to 0} \sqrt{\rho}\,\boldsymbol{G} \to \boldsymbol{UC} \tag{53}$$

where U is an $m \times m$ unitary matrix, i. e., $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}_m$. We consider the controller TFM, $\boldsymbol{K}(s)$ as

$$\boldsymbol{K}(s) = -\boldsymbol{G}(s\boldsymbol{I}-\boldsymbol{N}-\boldsymbol{BG}+\alpha^{-2}\boldsymbol{HC})^{-1}\boldsymbol{H} \tag{54}$$

If Re $\lambda_i(\boldsymbol{N}+\boldsymbol{BG})<0$, Re $\lambda_i(\boldsymbol{N}+\alpha^{-2}\boldsymbol{HC})<0$, and $\lim_{\rho \to 0}\sqrt{\rho}\,\boldsymbol{G} \to \boldsymbol{UC}$, then the limiting behavior of $\boldsymbol{K}(s)$ as $\rho \to 0$ is as follows (Han, 1995):

$$\begin{aligned}&\lim_{\rho \to 0}\boldsymbol{K}(s) \to [\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{N})\boldsymbol{B}]^{-1}[\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{N})^{-1}\boldsymbol{H}] \\ &= \boldsymbol{G}^{-1}(s) \cdot \boldsymbol{G}_{KF}(s)\end{aligned} \tag{55}$$

2.4 Modified Riccati equation for the compensated system

In order to calculate the statistically linearized plant, DF gains for nonlinearties should be assumed before they are calculated. Their exact values can be obtained by solving the modified Riccati equation of the closed-loop for the compensated system. The modified Riccati equation is given from Eq.

$$\tilde{\boldsymbol{N}}\tilde{\boldsymbol{Q}}_\infty + \tilde{\boldsymbol{Q}}_\infty\tilde{\boldsymbol{N}}^T + \gamma^{-2}\tilde{\boldsymbol{Q}}_\infty\tilde{\boldsymbol{R}}_\infty\tilde{\boldsymbol{Q}}_\infty + \tilde{\boldsymbol{V}} = \boldsymbol{0} \tag{56}$$

where,

$$\tilde{N}=\begin{bmatrix} N(\sigma_x) & BG \\ HC & N_z(\sigma_z) \end{bmatrix}$$

$$\tilde{R}_\infty=\begin{bmatrix} C^TC/\mu & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{V}=\begin{bmatrix} V_1 & 0 \\ 0 & HV_2H^T \end{bmatrix}$$

If we consider the nonlinear correction term in Eq. (20) in the cheap control problem, then we must solve Eqs. (20) and (56) simultaneously with respect to the guessed unknown variables [$\tilde{Q}_\infty$; $n(2n+1)$, $\bar{P}$; $n(n+1)/2$], where n is the number of plant states. Since it requires a great deal of computation time for a higher-order plant, it is very diffecult to find the solution. Fortunately, as a result of the cheap control problem as shown in Section 2.2, when the good LTR condition is satisfied, the nonlinear correction term Ψ in Eq. (20) can be neglected. Hence, the control gains and the stationary statistics of the compensated system can be separately calculated from the LTR procedure and modified Riccati equation.

2.5 Nonlinear controller synthesis using IRIDF techniques

The set of DF gains whose parameters depend on the stationary statistics of several input ranges is synthesized into a nonlinear unction that can be executed via IRIDF techniques. Atherton (Atherton, 1957) presented the theoretical explanation of IRIDF techniques. For practical application, however, approximated method suggested by Suzuki (Suzuki and Hedrick, 1985) is more convenient to synthesize DF into a nonlinear function. The approximation can be described roughly as follows. First, it is possible to resolve the given DF gain into several constituent gains:

$$N(\sigma_x)=N_1(\sigma_x)+N_2(\sigma_x)+\cdots+N_n(\sigma_x) \quad (57)$$

The corresponding inverse DF can be then written as

$$f(x)=f_1(x)+f_2(x)+\cdots+f_n(x) \quad (58)$$

where each $f_i(x)$ is the inverse DF of the corresponding $N_i(\sigma_x)$. It is thus more useful to factor the unknown gain into constituent gains whose inverse describing functions are either known or easy to determine. This can be done by iteratively until the desired level of accuracy has been reached. More detailed explanations about IRIDF techniques are presented in (Kim, 1995). And the design procedurs of the QLQG/H$_\infty$/LTR control system are the same as the QLQG/LTR case (Kim, 1989).

3. Design Example

3.1 Problem formulation

As a design example, we consider a flexible structure model with Coulomb friction (Kimura, Oike, and Miura, 1991). The nonliner flexible model is linearized via statistical linearization techniques. The transfer function $\tilde{G}(s)$ of the statistically linearized model is

$$\tilde{G}(s)=\frac{A_0}{s(s+N_0)}+\sum_{i=1}^{\infty}\frac{A_i}{s^2+(2\zeta_i\omega_i+N_i)s+\omega_i^2} \quad (59)$$

Numerical values of the parameters in (59) are shown in Table 1, and in addition $N_i=\left(\frac{T_{ci}\sqrt{2/\pi}}{\sigma_{xi}}\right)$ is the DF gain for Coulomb friction, σ_{xi} is the standard deviation for the each state, and T_{ci} is the magnitude of the load Coulomb friction. We take the rigid mode and the first two elastic modes as a nominal model. Then

$$G(s)=\frac{A_0}{s(s+N_0)}+\sum_{i=1}^{2}\frac{A_i}{s^2+(2\zeta_i\omega_i+N_i)s+\omega_i^2} \quad (60)$$

The residual modes constitute the perturbation.

$$\Delta G(s)=+\sum_{i=3}^{5}\frac{A_i}{s^2+(2\zeta_i\omega_i+N_i)s+\omega_i^2} \quad (61)$$

The design specifications considered are as follows:

Table 1 Numerical values of the model parameters

i	ω_i	$\zeta_i\omega_i$	A_i	T_{ci}
0			9.567	0.1
1	7.932	0.0397	11.99	0.01
2	25.12	0.1256	0.637	0.01
3	26.17	0.1309	0.812	0.01
4	74.34	0.3717	0.016	0.01
5	85.79	0.4289	4.144	0.01

1. Steady-state tracking errors should be zero for arbitrary constant inputs.
2. Gain crossover frequency should be about 1. 5 rad/sec.
3. The nominal controller should have stability-robustness to unstructured model errors.

We now, design three control systems, the LQG/H$_\infty$/LTR, QLQG/LTR and QLQG/H$_\infty$/LTR, and demonstrate that the proposed QLQG/H$_\infty$/LTR approach has good stability-robustness to unstructured model errors and nonlinear effects when compared with the other two approaches.

3.2 Linear controller design using the LQG/H$_\infty$/LTR method

We apply the LQG/H$_\infty$/LTR method for a linear plant where Coulomb friction ($T_{ci}\cdot sgn(x_i)$) is assumed to be linear (x_i). Then the design plant model is

$$G(s)=\frac{A_0}{s^2}+\sum_{i=1}^{2}\frac{A_i}{s^2+2\zeta_i\omega_i s+\omega_i^2} \tag{62}$$

The target filter loop is designed by matching the high-frequency singular values. Then the design parameter $\boldsymbol{L}$ is chosen as follows:

$$\boldsymbol{L}=\alpha^{-1}[\boldsymbol{C}^T(\boldsymbol{C}\boldsymbol{C}^T)^{-1}],\quad \alpha^{-2}=1-\gamma^{-2} \tag{63}$$

The values of the design parameters γ and μ are chosen as 1.01 and 0.05, respectively, to satisfy the design specifications. Then the filter gain matrix is calculated from Eqs. (19) and (29):

$$\boldsymbol{H}=[0.267\ \ -0.052\ \ 0.087\ \ -0.111\ \ 1.491\ \ 0]^T \tag{64}$$

LTR is attempted with the cheap control problem. The target filter loop is recovered up to a decade beyond the crossover frequency. For this level of recovery, we choose the value of ρ as 10^{-7}. Then the control gain is calculated from Eqs. (20) and (21) without the correction term, Eq. (31):

$$\boldsymbol{G}=[-1007.0\ \ -41.73\ \ -1002.1\ \ -41.66\ \ -1007.6\ \ -41.73]^T \tag{65}$$

We now check the stability-robustness of the designed LQG/H$_\infty$/LTR controller with nonlinear plant. We assume the white noise intensities of the selected disturbance inputs (V_2) are 5, 10^{-3} and 10^{-5} to represent the different input

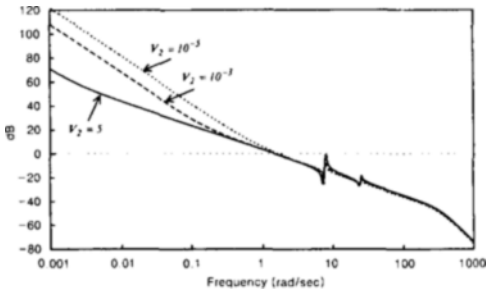

Fig. 2 Singular value plots of the loop transfer function of the nonlinear plant with the LQG/H$_\infty$/LTR controller

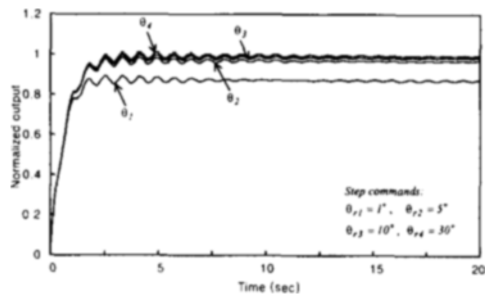

Fig. 3 Step responses of the LQG/H$_\infty$/LTR control system

cases. For the LQG/H$_\infty$/LTR controller with nonlinear plant, Figs. 2 and 3 show singular value plots of the loop transfer function and normalized step responess, respectively.

In Fig. 2, the singular value shapes of the loop transfer function change largely with the magnitude of the white noise intensities at low frequency. In Fig. 3, the nominal LQG/H$_\infty$/LTR control system does not exhibit the stability-robustness, and there are some steady-state errors for all constant inputs of the ranges of interest. These errors are due to the effect of Coulomb friction and the fact that the LQG/H$_\infty$/LTR controller cannot capture this nonlinear effect for large command inputs. Thus, a nonlinear controller is required to capture the effect of Coulomb friction and to adapt to changes in command input.

3.3 Nonlinear controller design using the QLQG/H$_\infty$/LTR and QLQG/LTR methods

We now design the QLQG/H$_\infty$/LTR controller with the same design parameters as for the LQG/

H_∞/LTR case. When the design parameter $\gamma \to \infty$, we can also design the QLQG/LTR controller with the same procedures as used for the QLQG/H_∞/LTR. We select the statistically linearized nominal plant as the design plant model. To cover operating range of interest, we choose zero-mean white noise intensities of disturbance inputs (V_2) between 5 and 10^{-5}. At each selected value of intensities, design procedures of the QLQG/H_∞/LTR are executed. We store the gains (filter, control and DF) and the stationary statistics (controller states). The filter and control gains are almost constant for any inputs (V_2) and therefore if any gain is selected, the effects of gain variations on the performance of the designed system are small. We select the values of the filter and control gain when the intensity of V_2 is 10^{-3}. The filter and control gains are obtained as

QLQG/H_∞/LTR:

$$\boldsymbol{H} = [0.017 \ \ -0.058 \ \ 0.067 \ \ -0.112 \ \ 1.491 \ \ 0]^T \tag{66}$$

$$\boldsymbol{G} = [-1006.3 \ \ -41.69 \ \ -1001.3 \ \ -41.65 \ \ -1006.8 \ \ -41.70] \tag{67}$$

QLQG/LTR($= \lim_{\gamma \to \infty}$ QLQG/H_∞/LTR):

$$\boldsymbol{H} = [1.392 \ \ -0.237 \ \ 1.254 \ \ -1.207 \ \ 1.491 \ \ 0]^T \tag{68}$$

$$\boldsymbol{G} = [-999.46 \ \ -41.57 \ \ -9945.3 \ \ -41.50 \ \ -1000.0 \ \ -41.57] \tag{69}$$

The DF gains (N_1, N_2 and N_3) and the standard deviations of the QLQG/H_∞/LTR case are given in Table 2. As N_1, N_2 and N_3 are function of z_2, z_4 and z_6, respectively, we can obtain the desired nonlinear function, which is the same as the QLQG/LTR case, via IRIDF techniques. Fig. 4 shows these results.

Figure 5 shows shapes of the recovered loop transfer function of both control systems given to input intensities of 5, 10^{-3} and 10^{-5}, respectively, to see the shapes for maximum, middle and minimum input values. The time responses for the different command input are given in Figs. 6

Table 2 DF gains and standard deviations of the controller states at selected operating values

V_2	5	1	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
N_1	0.002	0.0046	0.0151	0.0492	0.165	0.552	0.972
N_2	0.001	0.0022	0.0079	0.031	0.124	0.448	0.658
N_3	0.0008	0.0019	0.0064	0.021	0.074	0.276	0.416
σ_{z2}	37.94	16.995	5.30	1.622	0.483	0.114	0.101
σ_{z4}	5.836	2.527	0.720	0.181	0.054	0.0169	0.0117
σ_{z6}	7.149	3.185	0.994	0.303	0.090	0.0267	0.0181

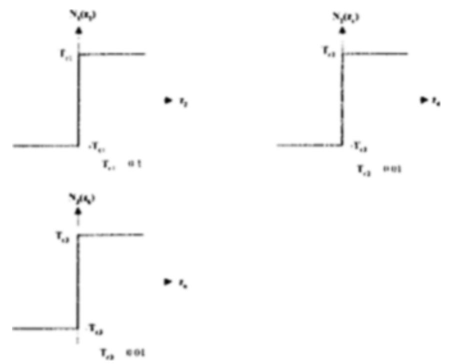

Fig. 4 Desired nonlinear function via the IRIDF techniques

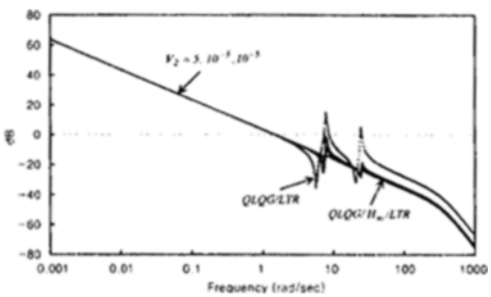

Fig. 5 Singular value plots of the loop transfer function of the QLQG/H_∞/LTR and QLQG/LTR control systems

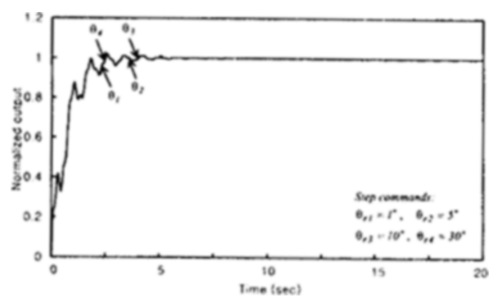

Fig. 6 Step responses of the QLQG/LTR control system

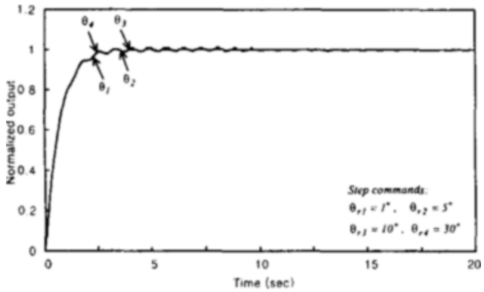

Fig. 7 Step responses of the QLQG/H∞/LTR control system

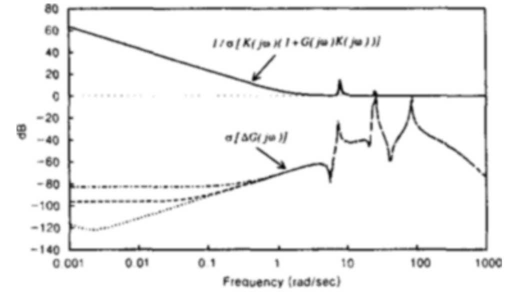

Fig. 8 Singular value plots for the $1/[K(s)(1+G(s)K(s))^{-1}]$ and $\Delta G(s)$ of the QLQG/LTR system

(QLQG/LTR) and 7 (QLQG/H∞/LTR). When we compare these results with those of the linear control approach (LQG/H∞/LTR), we see that nonlinear control approaches (QLQG/H∞/LTR and QLQG/LTR) are insensitive to the effects of Coulomb friction. Thus, for both nonlinear controllers with nominal plant, the stability robustness for nonlinear effects is maintained and steady state tracking errors do not exist.

3.4 Comparison of the QLQG/H∞/LTR and QLQG/LTR controller

The QLQG/LTR approach, apart from the fact that it can treat hard nonlinearities, has similar performance Characteristics with the LQG/LTR design. For the same reason, the proposed QLQG/H∞/LTR design has similar performance characteristics with the linear mixed H_2 and H_∞ control (Bernstein, 1989), but the linear control method cannot address hard nonlinearities. The main advantage of the QLQG/H∞/LTR to the QLQG/LTR is that the QLQG/H∞/LTR system is more robust to unstructured model errors such as neglected higher elastic modes, etc. To show this, we check stability-robustness for model perturbations of both control systems. For the perturbed value $\Delta G(s)$ of the nominal plant $G(s)$, we consider $\tilde{G}(s)(=G(s)+\Delta G(s))$ as the true model in order to analyze additive uncertainty. The Small Gain Theorem gives the following sufficiency test for stability-robustness with additive uncertainty:

$$\bar{\sigma}(\Delta G(s)) < \frac{1}{\bar{\sigma}[K(s)(1+G(s)K(s))^{-1}]} \quad \forall \omega \qquad (70)$$

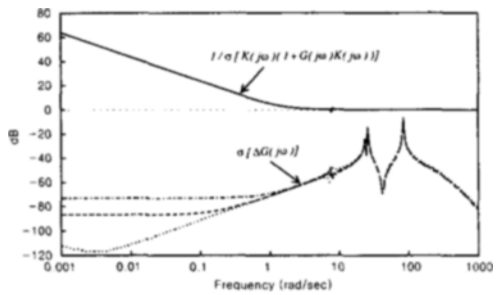

Fig. 9 Singular value plots for the $1/[K(s)(1+G(s)K(s))^{-1}]$ and $\Delta G(s)$ of the QLQG/H∞/LTR system

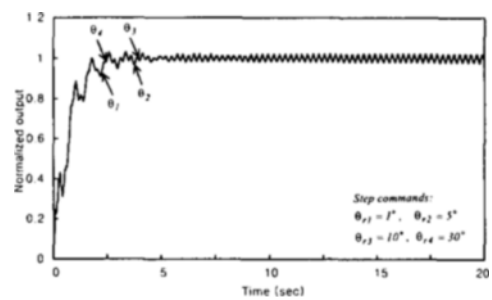

Fig. 10 Step responses of the QLQG/LTR controller with full model

By Condition (70), frequency domain performances of each system are shown in Figs. 8 and 9.

From these results, we can see that the QLQG/H∞/LTR system has better frequency domain performance than the QLQG/LTR system. To test the time-domain performance, we combine both the designed nonlinear controller with the full model and check its robustness by simulations.

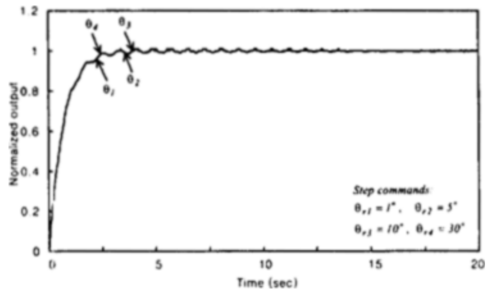

Fig. 11 Step responses of the QLQG/H_∞/LTR controller with full model

In Fig. 10, the neglected flexible modes destabilize the nominal QLQG/LTR system. In contrast to this result, Fig. 11 shows that the QLQG/H_∞/LTR system is a better design method than the QLQG/LTR with respect to robustness to unstructured model uncertainties.

4. Conclusion

We have presented a nonlinear QLQG/H_∞/LTR multivariable design method. For a multivariable system with hard nonlinearties such as Coulomb friction, dead-zone and backlash, the suggested method allows one to design a nonlinear controller systematically. The design is an integration of statistical linearization, mixed LQG and H_∞ optimization, modified loop shaping and loop transfer recovery techniques. The nonlinear effects are considered in the nonlinear controller by IRIDF techniques, which can adapt the different input values.

The modified LTR conditions for the statistically linearized system were discussed. The mixed LQG and H_∞ optimization process for the statistically linearized system yields the nonlinear correction term in the modified CARE. Fortunately, it is found that the nonlinear correction term is not dominant if good LTR is satisfied. In the case the modified CARE is the same as the linear one of the mixed LQG and H_∞ control. The QLQG/LTR design is also a nonlinear controller design method for hard nonlinear systems but it considers design performance from the viewpoint of the LQG optimality criterion. For unstructured system uncertainties, our QLQG/H_∞/LTR method has good robustness compared with the QLQG/LTR, since our approach comes from mixed LQG(H_2) and H_∞ optimal processes. The suggested control method is a general version of the QLQG/LTR system. That is, when the system parameter $\gamma \rightarrow \infty$, the QLQG/$H_\infty$/LTR is identical to QLQG/LTR.

Finally, we have applied the LQG/H_∞/LTR, QLQG/H_∞/LTR and QLQG/LTR methods to a flexible link system with Coulomb friction. We have shown that both nonlinear controllers are insensitive to the nonlinear effects of the given nonlinear model, but that the suggested QLQG/H_∞/LTR controller is more robust to unstructured uncertainty than QLQG/LTR.

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